1. Combinatorics

Q.1.2.1.1 Count the number of ways the letters in the word *token* can be arranged, so that there is no repetition.
Answer: The letters in *token* have to be placed without any repetition. 1st place can be filled in 5 ways; 2nd place can be filled in 4 ways; 3rd place can be filled in 3 ways; 4th place can be filled by 2 ways; 5th place can be filled by 1 way. Using multiplication rule, total number of arrangements (without repetition) $= 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Q.1.2.1.2 How many ways can you get a sum of 4 or 12 using two identifiable dice?
Answer: Let $(i,\ j)$ be the order pair, where i spots appear in the first die, and j spots appear in the second die. Then the favourable cases for getting sum of 4 are (1, 3), (3, 1), and (2, 2). The number of such cases is equal to 3. Again, the favourable case for getting sum of 12 is (6, 6). The number of cases is equal to 1. Thus, the total number of cases of getting a sum of 4 or 12 is equal to $3 + 1 = 4$.

Q.1.2.1.3 Find the coefficient of x^5 in $(1 + 2x - x^2)^7$.
Answer: General term is $= \frac{7!}{n_1!n_2!n_3!}(1)^{n_1}(2x)^{n_2}((-x)^2)^{n_3}$
$= \frac{7!2^{n_2}(-1)^{n_3}}{n_1!n_2!n_3!}x^{n_2+2n_3}$

$$n_1 + n_2 + n_3 = 7 \tag{1}$$

$$n_2 + 2n_3 = 5 \tag{2}$$

Values of n_1, n_2 and n_3 that satisfy (1) and (2) are given as follows.

Table 1. Possible values of n_1, n_2, n_3

n_1	n_2	n_3
2	5	0
3	3	1
4	1	2

The corresponding terms are $\frac{7!2^5(-1)^0}{2!5!0!}x^5$, $\frac{7!2^3(-1)^1}{3!3!1!}x^5$, and $\frac{7!2^1(-1)^2}{4!1!2!}x^5$. Therefore, the coefficient of x^5 in the given expression
$= \frac{7!32}{2!5!} - \frac{7!8}{3!3!} + \frac{7!2}{4!2!} = -238$.

Q.1.2.1.4 How many ways a word of 3 letters can be formed from the word *token*?

Answer: 3 letters can be chosen from *token* in ${}^5C_3$ ways. 3 letters in a word can be arranged in 3! ways. Thus, the number of ways a word of 3 letters can be formed is $= {}^5C_3 \times 3! = \frac{5!3!}{3!(5-3)!} = \frac{5!}{2!} = 60$.

Q.1.2.1.5 Three-digit numbers are formed from the set {0, 1, ..., 9} using (i) with repetition, (ii) without repetition. Find the total possible numbers in each case.
Answer: (i) Each of the three places can be filled in one of 10 digits, i.e., 10 possible ways. So, the total number of ways it can be done is $= 10 \times 10 \times 10 = 1000$.
(ii) First place can be filled in 10 ways. Second place can be filled in 9 ways. Third place can be filled in 8 ways. The total number of ways it can be done is $= 10 \times 9 \times 8 = 720$.

Q.1.2.1.6 Show that the number of circular permutations is (n-1)! for n objects.
Answer: Let the objects be $a_1, a_2, \ldots, a_n$. We shall prove it using the method of induction. If there are 2 objects, the possible circular permutations is a_1a_2. Permutations a_1a_2 and a_2a_1 are essentially same, when objects a_1 and a_2 are placed in a circular manner. Hence, the number of circular permutation is 1. So, the result is true for $n = 2$.
Let the result be true for $n = k$. In this case, the number of circular permutations is (k-1)!. Let us consider a particular circular permutation $a_1a_2 \ldots a_k$. Between a_ia_{i+1} or a_ka_1, a_{k+1} can be placed, $i = 1, 2, \ldots,$ k-1. There are k places for each circular permutation. Thus, the total number of permutations for $(k+1)$ objects is $k.(k-1)! = k!$. The result is true for $n = k + 1$.

Q.1.2.1.7 Find the number of ways 5 men and 5 women sit around a table so that no two women sit together.
Answer: Five men can sit around a table in (5 - 1)! $= 4! = 24$ ways. In the round table there is a seat, one between every pair of men. These 5 seats can be occupied by 5 women in 5! ways. Then the total number of ways it can be done is equal to $24 \times 5! = 2880$.

Q.1.2.1.8 How many ways can one arrange 7 different beads to form a necklace.
Answer: 7 different beads can be arranged in a circular manner in $(7 - 1)! = 6!$ ways. Here, there is no distinction between clockwise and anticlockwise arrangements. So, the required number of distinct ar-

rangements is equal to $\frac{1}{2} \times 6! = 360$.

Q.1.2.1.9 There are 8 people with 4 men and 4 women.
(i) Find the number of ways a committee of 5 people can be formed.
(ii) How many ways a committee be formed such that all 4 women are available in the committee along with 2 men?
(iii) A committee of 2 people is required to form so that there is a person from each gender.
Answer: (i) 5 people can be selected from 8 people in ${}^8C_5$ ways $= 56$ ways.
(ii) All 4 women can be selected in ${}^4C_4$ ways. 2 men can be selected in ${}^4C_2$ ways.
Then, total number of committees is equal to ${}^4C_4 \times {}^4C_2 = 6$.
(iii) 1 man can be selected in ${}^4C_1$ ways. 1 woman can be selected in ${}^4C_1$ ways. Thus, the total number of two member committees is equal to ${}^4C_1 \times {}^4C_1 = 16$.

Q.1.2.1.10 Find the number of different license plates if each plate consists of two letters followed by two digits and then one letter followed by four digits. (An example: GA-05-B-3368)
Answer: First two letters can be filled in $26 \times 26 = 26^2$ ways. The following two digits can be filled in $10 \times 10 = 10^2$ ways. Then a single letter can be filled in 26 ways. The remaining part, i.e. 4 digits, can be chosen in $10 \times 10 \times 10 \times 10 = 10^4$ ways. The total number of license plates is equal to $26^2 \times 10^2 \times 26 \times 10^4 = 10^6 \times 26^3$.

Q.1.2.1.11 There are 4 lists of projects containing 11 projects, 20 projects, 9 projects and 10 projects. How many ways can two students choose two projects such that there is no repetition.
Answer: Total number of projects $= 11 + 20 + 9 + 10 = 50$. First student can choose any one of 50 projects. The second student can choose any one of remaining 49 projects. Total number of ways two projects can be chosen $= 50 \times 49 = 2450$.

Q.1.2.1.12 Let a password be of at least six characters long but at the most eight characters long having at least one digit. The character set is $\{a, \ldots, z, 0, \ldots, 9\}$. Find the number of possible passwords.
Answer: Let T_i be the number of possible passwords of length i using atleast 1 digit, $i = 6, 7, 8$. Total number of characters $= 26 + 10 = 36$.
$T_i = 36^i - 26^i, i = 6, 7, 8$.

2. Mathematical Induction

Q.1.2.2.1 Prove that $n! \geq 2^{n-1}$, for $n = 1, 2, 3, \ldots$.
Answer: We shall prove the result using the method of induction.
Basis step: For $n = 1$, $1! = 1$ and $2^{1-1} = 2^0 = 1$. So, $1! \geq 2^{1-1}$.
Induction hypothesis: Assume that $i! \geq 2^{i-1}$, for $i = 1, 2, \ldots, n$.
Induction step: $(n+1)! = (n+1)n! \geq (n+1)2^{n-1}$, by induction hypothesis
Thus, $(n+1)! \geq 2.2^{n-1}$, since $(n+1) \geq 2$
$\Rightarrow (n+1)! \geq 2^{(n+1)-1}$
So, the result is true for $i = n + 1$.

Q.1.2.2.2 Define $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$, for $k \geq 1$. Prove that $H_{2^n} \geq 1 + \frac{n}{2}$, for $n \geq 0$ using the method of induction.
Answer: Basis step: For $n = 0$, $H_{2^0} = 1 \geq 1 = 1 + \frac{0}{2}$.
Induction hypothesis: Assume that $H_{2^i} \geq 1 + \frac{i}{2}$ for $i = 0, 1, 2, \ldots, n$.
Induction step: $H_{2^{n+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}}$.
$H_{2^{n+1}} = H_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n}$, since $2^n + 2^n = 2.2^n = 2^{n+1}$
$\geq 1 + \frac{n}{2} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n}$, using induction hypothesis
$\geq 1 + \frac{n}{2} + 2^n.\frac{1}{2^n+2^n} = 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}$. It is true for $i = n + 1$.

Q.1.2.2.3 Show that $\frac{1}{2n} \leq \frac{1.3.5...(2n-1)}{2.4.6...(2n)} \leq \frac{1}{\sqrt{n+1}}, n = 1, 2, 3, \ldots$
Answer: First, we shall prove the inequality $\frac{1}{2n} \leq \frac{1.3.5...(2n-1)}{2.4.6...(2n)}$ using the method of induction.
Basis step: $\frac{1}{2n} = \frac{1}{2.1} = \frac{1}{2} \leq \frac{1}{2}$
Induction hypothesis: Assume that the result is true for $n = k$.
Induction step: We shall prove that the result is true for $n = k + 1$.
$\frac{1.3.5...(2k-1)(2k+1)}{2.4.6...(2k)(2k+2)} \geq \frac{1}{2k}.\frac{2k+1}{2k+2}$ [by induction hypothesis]
$= \frac{2k+1}{2k}.\frac{1}{2k+2} \geq \frac{1}{2k+2}$
The result is true for $n = k + 1$.
Now, we shall prove the inequality $\frac{1.3.5...(2n-1)}{2.4.6...(2n)} \leq \frac{1}{\sqrt{n+1}}, n = 1, 2, 3, \ldots,$ using the method of induction.
For $n = 1, \frac{1}{2} \leq \frac{1}{\sqrt{2}}$, since $2 \geq \sqrt{2}$
Let it be true for $n = k$. Therefore, $\frac{1.3.5...(2k-1)}{2.4.6...(2k)} \leq \frac{1}{\sqrt{k+1}}, n = 1, 2, 3, \ldots$
Consider $n = k + 1$. $\frac{1.3.5...(2k-1)(2k+1)}{2.4.6...(2k)(2k+2)} \leq \frac{1}{\sqrt{k+1}}.\frac{2k+1}{2k+2}$, by induction hypothesis

To show $\frac{1}{\sqrt{k+1}}.\frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{k+2}}$, it is enough to show $\frac{k+2}{k+1} \leq (\frac{2k+2}{2k+1})^2$
i.e., to show $1 + \frac{1}{k+1} \leq (1 + \frac{1}{2k+1})^2 = 1 + \frac{2}{2k+1} + \frac{1}{(2k+1)^2}$
i.e., to show $0 \leq \frac{2}{2k+1} - \frac{1}{k+1} + \frac{1}{(2k+1)^2} = \frac{1}{(2k+1)(k+1)} + \frac{1}{(2k+1)^2}$
Now, for some integer $k \geq 0$, the expression $\frac{1}{(2k+1)(k+1)} + \frac{1}{(2k+1)^2} \geq 0$
This follows the induction step.

Q.1.2.2.4 What is pigeonhole principle?
Answer: If A and B are nonempty finite sets and $|A| > |B|$, then there is no one-to-one function from A to B. In otherwords, if we attempt to pair off the elements of A (the "pigeons") with elements of B (the "pigeonholes"), sooner or later we will have to put more than one pigeon in a pigeonhole.

Q.1.2.2.5 Show by induction that $n^4 - 4n^2$ is divisible by 3, when $n(\geq 0)$ is an integer.
Answer: Let $f(n) = n^4 - 4n^2$. $f(0) = 0^4 - 4.0^2 = 0$, and it is divisible by 3.
Assume that $f(n) = n^4 - 4n^2$ is divisible by 3, for $n = k$.
i.e., we assume $f(k) = k^4 - 4k^2$ is divisible by 3.
We have to prove that $f(k+1) = (k+1)^4 - 4(k+1)^2$ is divisible by 3.
Now, $f(k+1) - f(k) = (k+1)^4 - 4(k+1)^2 - k^4 + 4.k^2$
$= 4k^3 + 6k^2 - 4k - 3 = 4k(k^2 - 1) + 3(2k^2 - 1)$
$= (k^2 - 1)(4k + 3) + 3k^2 = (k^2 - 1)(3k + 3) + 3k^2 + k(k^2 - 1)$
$= 3\{k^2 + (k+1)(k^2 - 1)\} + (k - 1)k(k + 1) = t_1 + t_2$
where, $t_1 = 3\{k^2+(k+1)(k^2-1)\}$ is divisible by 3, and $t_2 = (k-1)k(k+1)$ is a product of three consecutive integers.
So, it is divided by 3. Then, $f(k+1) - f(k) = t_1 + t_2$ is divisible by 3.
Thus, if $f(k)$ is divisible by 3, then $f(k+1) = f(k) + t_1 + t_2$ is also divisible by 3.
Therefore, $f(n) = n^4 - 4n^2$ is divisible by 3, when $n(\geq 0)$ is an integer.

Q.1.2.2.6 Prove by induction $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.
Answer: Let $f(n) = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \frac{1}{(2n-1)(2n+1)}$
Now, $f(1) = \frac{1}{1.3} = \frac{1}{3} = \frac{1}{2.1+1}$. So, it is true for $n = 1$.
Assume that it is true for $n = k$.
$f(k) = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$
Now, $f(k+1) = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$
$= f(k) + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{1}{(2k+1)}[k + \frac{1}{(2k+3)}]$
$= \frac{(2k+1)(k+1)}{(2k+1)\{2(k+1)+1\}} = \frac{k+1}{2(k+1)+1}$
It is true for $n = k + 1$.
Thus, $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}, \forall n \geq 1$.

Q.1.2.2.7 Prove by induction the following inequality: $n < 2^n, n = 1, 2, 3, \ldots$
Answer: For $n = 1, 1 < 2$, i.e. $1 < 2^1$. Thus, it is true for $n = 1$.
Let it be true for $n = k$. Then $k < 2^k$ (induction hypothesis)
So, $k+1 < 2^k+1 < 2^k+2^k = 2.2^k = 2^{k+1}$. Thus, it is true for $n = k+1$.

Q.1.2.2.8 Consider harmonic numbers as defined below.
$H_i = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}, i = 1, 2, 3, \ldots$
Show that $H_{2^n} \geq 1 + \frac{n}{2}, n = 0, 1, 2, \ldots$ (Use mathematical induction)
Answer: For $n = 0, H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$. The result is true for $n = 0$.
Let it be true for $n = k$. Then, $H_{2^k} \geq 1 + \frac{k}{2}$.
$H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}$
$= H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}$
$\geq (1 + \frac{k}{2}) + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}$
$\geq (1 + \frac{k}{2}) + 2^k.\frac{1}{2^{k+1}}$
$= 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}$
It is true for $n = k + 1$, and the result follows.

Q.1.2.2.9 Apply mathematical induction to prove that
$2 - 2.7 + 2.7^2 - \cdots + 2(-7)^n = \frac{(1-(-7)^{n+1})}{4}, n = 0, 1, 2, \ldots$
Answer: For, $n = 0$, LHS = 2, RHS = $\frac{(1-(-7)^1)}{4} = \frac{8}{4} = 2$
Therefore, the result is true for $n = 0$.
Let the result be true for $n = k$ (induction hypothesis)
For $n = k + 1$, LHS = $2 - 2.7 + 2.7^2 - \cdots + 2(-7)^k + 2.(-7)^{k+1}$
$= \frac{(1-(-7)^{k+1})}{4} + 2.(-7)^{k+1}$ (by induction)
$= \frac{1}{4} - \frac{(-7)^{k+1}}{4} + \frac{8}{4}.(-7)^{k+1}$
$= \frac{1+7.(-7)^{k+1}}{4} = \frac{1-(-7)(-7)^{k+1}}{4} = \frac{1-(-7)^{k+2}}{4} =$ RHS
The result is true for $n = k + 1$.

Q.1.2.2.10 Show that $H_1 + H_2 + \cdots + H_n = (n + 1)H_n - n$, where $H_i = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}, i = 1, 2, 3, \ldots$
Answer: We apply here the method of mathematical induction. For $n = 1$, LHS = 1, RHS = (1 + 1) 1 - 1. So, it is true for $n = 1$.

3. Recurrence Relation

Q.1.2.3.1 Let $C_1 = 1$ and let $C_n = C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1$, for $n > 1$. Determine the final five values of C_n.
Answer: $C_2 = C_1C_1 = 1.1 = 1, C_3 = C_1C_2 + C_2C_1 = 1.1 + 1.1 = 2$
$C_4 = C_1C_3 + C_2C_2 + C_3C_1 = 1.2 + 1.1 + 2.1 = 5$
$C_5 = C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 = 1.5 + 1.2 + 2.1 + 5.1 = 14$

Q.1.2.3.2 Let H_n be n-th harmonic number. Show that $H_n \leq \frac{n+1}{2}$.
Answer: The following recurrence relation for a sequence is known as harmonic numbers.
$H_1 = 1$ and for $n > 1$ let $H_n = H_{n-1} + \frac{1}{n}$
$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + (n-1).\frac{1}{2} = \frac{n+1}{2}$

Q.1.2.3.3 Find the recurrence relation that is formed by the sequence $a_n = n^2 - 6n + 8$.
Answer: $a_n = n^2 - 6n + 8,\ a_{n-1} = (n-1)^2 - 6(n-1) + 8$
Thus, $a_n - a_{n-1} = 2n + 5$.

Q.1.2.3.4 Solve the linear homogeneous recurrence relation with constant coefficients.
$a_n = a_{n-1} + a_{n+1}, n > 1$ (1)
where, $a_0 = 0$ and $a_1 = 1$ (2)
Answer: The characteristic equation of (1) is $x^2 - x - 1 = 0$. It has characteristic roots $\phi = \frac{(1+\sqrt{5})}{2}$ and $\phi' = \frac{(1-\sqrt{5})}{2}$. So, the general solution of (1) is
$a_n = C_1(\frac{1+\sqrt{5}}{2})^n + C_2(\frac{1-\sqrt{5}}{2})^n, C_1$ and C_2 are constants.
$a_0 = C_1 + C_2 = 0$ (using (2)) (3)
$a_1 = C_1(\frac{1+\sqrt{5}}{2}) + C_2(\frac{1-\sqrt{5}}{2}) = 1$ (using (2)) (4)
By solving (3) and (4), we get $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$
The general solution of (1) becomes $a_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$.

Q.1.2.3.5 If c and d are constants with $d > 1$ and $a_n \leq da_{\lfloor \frac{n}{d} \rfloor} + cn$ then $a_n \leq cnlog_d(n) + a_1n$.
Answer: We shall prove this by induction on n. For the base case, we have that $a_1 \leq 0 + a_1.1$. We assume that the theorem is true for all $n < k$ and we examine a_k.

$a_k \leq da_{\lfloor \frac{k}{d} \rfloor} + ck$ (by the given condition)
$\leq d[c\lfloor \frac{k}{d} \rfloor log_d(\lfloor \frac{k}{d} \rfloor) + a_1\lfloor \frac{k}{d} \rfloor] + ck$ (by induction hypothesis)
$\leq d[c(\frac{k}{d})log_d(\frac{k}{d}) + a_1(\frac{k}{d})] + ck$ (since $\lfloor \frac{k}{d} \rfloor \leq \frac{k}{d}$)
$= cklog_d(\frac{k}{d}) + a_1k + ck = ck[log_d(k) - 1] + a_1k + ck$
$= cklog_d(k) + a_1k$

Q.1.2.3.6 Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3n$ with $a_0 = 1$ and $a_1 = 3$.
Answer: $a_n = 4a_{n-1} - 4a_{n-2} + 3n$ (5)
The homogeneous equation of (5) is $a_n - 4a_{n-1} + 4a_{n-2}$ (6)
The characteristic polynomial of (6) is $x^2 - 4x + 4 = 0$, or $(x-2)^2 = 0$.
The characteristic roots of (6) are $x_1 = 2$ and $x_2 = 2$.
General solution of (6) is $a_n = (k_1 + k_2n)2^n$, where k_1 and k_2 are constants. (7)
Since non-homogeneous part is a polynomial in n of degree 1, so the particular solution of (5) is also a polynomial in n of degree 1.
Let $a_n = k_3 + k_4n$ be a particular solution. So, it satisfies (5).
$k_3 + k_4n = 4(k_3 + k_4(n-1)) - (k_3 + k_4(n-2)) + 3n$
or, $k_3 + k_4n = 4nk_4 - 4k_4 - 4nk_4 + 8k_4 + 3n$
o,r $k_3 + k_4n = 4k_4 + 3n$
Equating the coefficients of n^1, n^0 in both sides, we get
$k_3 = 4k_4$ and $k_4 = 3$
$k_3 = 4k_4 = 4.3 = 12$.
Particular solution of (5) is $a_n = 12 + 3n$.
General solution of (5) is $a_n = (k_1 + k_2n)2^n + 12 + 3n$.
$a_0 = 1 \Rightarrow k_1 + 12 = 1$ or $k_1 = -11$
$a_1 = 3 \Rightarrow (k_1 + k_2).2 + 12 + 3 = 3$ or, $k_2 = 5$
$a_n = (-11 + 5n)2^n + 12 + 3n$ is the solution of (5).

Q.1.2.3.7 Solve $a_n = 2a_{n-1} + 3n^2 + 2.3^n$, where $a_0 = 1$.
Answer: $a_n = 2a_{n-1} + 3n^2 + 2.3^n$ (8)
The homogeneous equation of (8) is $a_n - 2a_{n-1} = 0$ (9)
The characteristic polynomial of (9) is $x - 2 = 0$
The characteristic roots of (9) is $x_1 = 2$
The general solution of (9) is $a_n = k.2^n, k$ is a constant
Note that the non-linear part of (8) is a combination of a polynomial of degree 2 and an exponential function. So, the particular solution of (8) will be a combination of second degree polynomial and a_n similar exponential function.

Let $a_n = k_0 + k_1 n + k_2 n^2 + k_3 3^n, k_i$ is a constant, $i = 0, 1, 2, 3, \ldots$ be a solution. So, it satisfies (8).
$k_0 + k_1 n + k_2 n^2 + k_3 3^n = 2(k_0 + k_1 n - k_1 + k_2 n^2 - 2k_2 n + k_2 + k_3.3^{n-1}) + 3n^2 + 2.3^n$
Equating the constant term, we get
$k_0 = 2k_0 - 2k_1 + 2k_2$ or, $k_0 - 2k_1 + 2k_2 = 0$ (10)
Equating the coefficient of n, we get $k_1 = 2k_1 - 4k_2$ or, $k_1 = 4k_2$. (11)
Equating the coefficient of n^2, we get $k_2 = 2k_2 + 3$ or, $k_2 = -3$. (12)
Equating the coefficient of 3^n, we have $k_3 = \frac{2k_3}{3} + 2$ or, $k_3 = 6$ (13)
Solving [using (10), (11), (12) and (13)] we get
$k_0 = -18,\ k_1 = -12,\ k_2 = -3$ and $k_3 = 6$.
General solution of (8) is $a_n = k.2^n - 18 - 12n - 3n^2 + 6.3^n$.
Given that $a_0 = 1$. So, $k - 18 + 6 = 1$ or, k=13.
$a_n = 13.2^n + 6.3^n - 3n^2 - 12n - 18$.

Q.1.2.3.8 Solve using generating function $a_n = 2a_{n-1} + 7$, where $a_0 = 0$.
Answer: Here, $a_n = 2a_{n-1} + 7$
or, $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2a_{n-1} x^n + \sum_{n=1}^{\infty} 7x^n, |x| < 1$
or, $G(x) - a_0 = 2xG(x) + 7x(1-x)^{-1}$, where, $G(x) = \sum_{n=0}^{\infty} a_n x^n$
or, $G(x) = 7x(1-x)^{-1}(1-2x)^{-1}$
or, $G(x) = 7x(1 + x + x^2 + \ldots)(1 + 2x + 2^2x^2 + \ldots)$
$a_n =$ co-efficient of $x^n = 7(2^{n-1} + 2^{n-2} + \ldots) = 7(2^n - 1)$.

Q.1.2.3.9 Solve using generating function.
$a_n = a_{n-1} + a_{n-2}, a_0 = a_1 = 1$
Answer: Now, $a_n = a_{n-1} + a_{n-2}$
or, $\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n$
or, $G(x) - a_0 - a_1 x = x(G(x) - a_0) + x^2 G(x)$, where, $G(x) = \sum_{n=0}^{\infty} a_n x^n$
$G(x) - x = xG(x) + x^2 G(x)$
or, $G(x)(1 - x - x^2) = x$
or, $G(x) = x(1 - x - x^2)^{-1}$
Now, $\frac{1}{1-x-x^2} = -\frac{1}{x^2+x-1} = -\frac{1}{(x-\alpha)(x-\beta)}$
We find the roots of equation: $x^2 + x - 1 = 0$
The roots are $x = \frac{-1\pm\sqrt{5}}{2}$
So, $\alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2}$
Now, $\frac{1}{(x-\alpha)(x-\beta)} = \frac{1}{(\alpha-\beta)}[\frac{1}{x-\alpha} - \frac{1}{x-\beta}]$
$G(x) = -\frac{x}{\sqrt{5}}[\frac{1}{x-\alpha} - \frac{1}{x-\beta}] = -\frac{x}{\sqrt{5}}[-(x-\alpha)^{-1} + (x-\beta)^{-1}]$
$G(x) = \frac{x}{\sqrt{5}}[\frac{1}{\alpha}(1 - \frac{x}{\alpha})^{-1} - \frac{1}{\beta}(1 - \frac{x}{\beta})^{-1}]$
$a_n = \frac{1}{\sqrt{5}}[\frac{1}{\alpha}.\frac{1}{\alpha^{n-1}} - \frac{1}{\beta}.\frac{1}{\beta^{n-1}}] = \frac{1}{\sqrt{5}}(\frac{1}{\alpha^n} - \frac{1}{\beta^n})$